

An Itzykson-Zuber-like Integral and Diffusion for Complex Ordinary and Supermatrices

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We compute an analogue of the Itzykson-Zuber integral for the case of arbitrary complex matrices. The calculation is done for both ordinary and supermatrices by transferring the Itzykson-Zuber diffusion equation method to the space of arbitrary complex matrices. The integral is of interest for applications in Quantum Chromodynamics and the theory of two-dimensional Quantum Gravity.

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I. INTRODUCTION

In 1980, Itzykson and Zuber¹ presented their result on a certain integral over the unitary group which had great impact in several areas of mathematical physics. Let U be a matrix parameterizing the unitary group $U(N)$ with the invariant Haar measure $d\mu(U)$. Moreover, consider two diagonal matrices x and y with entries x_n and y_n , respectively, where $n = 1, \dots, N$. The Itzykson-Zuber integral can then be written in the form

$$\int d\mu(U) \exp(i \operatorname{tr} U^{-1} x U y) = \frac{\det [\exp(ix_n y_m)]_{n,m=1,\dots,N}}{\Delta_N(x)\Delta_N(y)}, \quad (1.1)$$

where the function

$$\Delta_N(x) = \prod_{n < m} (x_n - x_m) \quad (1.2)$$

is the Vandermonde determinant of order N . Although it was later realized that this formula is a special case of a more general result due to Harish-Chandra,² it prompted many investigations in various fields. In 1983, Mehta and Pandey^{3,4} used this formula to work out, in the framework of Random Matrix Theory, the spectral correlations of a generic quantum chaotic system which undergoes a transition from conserved to broken time-reversal invariance. There are also numerous applications in field theory, particularly in the theory of two-dimensional Quantum Gravity; a review can be found in Ref. 5.

Recently, Shatashvili⁶ showed that the integral (1.1) itself and all correlations in the Itzykson-Zuber model can be evaluated using the Gelfand-Tzetlin coordinates for an explicit calculation. Remarkably, Itzykson and Zuber had not derived their result by an explicit calculation but related the integral (1.1) to a diffusion process. They showed that it can be viewed as the kernel of a diffusion equation in the curved space of the eigenvalues of Hermitean matrices. Since the space of Hermitean matrices is Cartesian and, therefore, easy to treat, the result (1.1) can be found by comparison with the curved space without actually calculating it explicitly. The crucial point is the separability of the Laplacian operator in the curved space of the eigenvalues. This diffusion equation technique turned out to be a very powerful tool.

Some years ago, it was realized by one of the present authors⁷ that the Itzykson-Zuber diffusion can be directly transferred to supermathematics. After the pioneering mathematical achievements of Berezin,⁸ supermathematics was brought into the theory of matrix models by Efetov⁹ and Verbaarschot, Weidenmüller, and Zirnbauer.¹⁰ In Ref. 7 it was shown that there is, completely analogously to the case of ordinary matrices, a diffusion equation in the space of the eigenvalues of Hermitean supermatrices whose kernel is the supersymmetric generalization of the Itzykson-Zuber integral (1.1). Thus, the integral can be worked out generalizing the methods of Ref. 1. Again, the crucial point is the separability of the Laplacian in the curved space of the eigenvalues.⁷ The result has been used to evaluate, in the framework of Random Matrix Theory, the effect of symmetry breaking on the spectral correlations of a chaotic

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time-reversal non-invariant system.¹¹ Recently, it was shown that the supersymmetric generalization of the Itzykson-Zuber diffusion has not only a mathematical, but also a physical meaning: in Random Matrix Theory, it describes the transition from arbitrary to chaotic fluctuations.¹² These results could be used to work out the crossover from Poisson regularity to chaos in a time-reversal non-invariant system.

Let u be a supermatrix, parameterizing the unitary supergroup $U(k_1/k_2)$, with invariant Haar-Berezin measure $d\mu(u)$, and let s and r be diagonal matrices, both having the form $s = \text{diag}(s_1, is_2)$ with $s_j = \text{diag}(s_{1j}, \dots, s_{kj})$, $j = 1, 2$. Then, in its most general form, the supersymmetric Itzykson-Zuber integral⁷ can be written as^{13,14}

$$\int d\mu(u) \exp(i\text{trg } usu^{-1}r) = \frac{\det[\exp(is_{p1}r_{p'1})]_{p,p'=1,\dots,k_1} \det[\exp(is_{q2}r_{q'2})]_{q,q'=1,\dots,k_2}}{B_{k_1 k_2}(s) B_{k_1 k_2}(r)}, \quad (1.3)$$

where the symbol trg stands for the supertrace, often also denoted by str . The supersymmetric generalization of the Vandermonde determinant is given by

$$B_{k_1 k_2}(s) = \frac{\Delta_{k_1}(s_1)\Delta_{k_2}(is_2)}{\prod_{p,q} (s_{p1} - is_{q2})}. \quad (1.4)$$

It is worth mentioning that $B_{k_1 k_2}(s)$ reduces to a determinant⁷ via Cauchy's lemma in the case $k_1 = k_2$ such that $B_{k_1 k_1}(s) = \det[1/(s_{p1} - is_{q2})]_{p,q=1,\dots,k_1}$. Furthermore, by setting $k_1 = N$ and $k_2 = 0$, one easily sees that formula (1.3) includes formula (1.1) as desired. However, for k_1 and k_2 both non-zero, there is an important caveat: In superanalysis, a change of variables can induce a certain class of singularities. Here this implies that, if further integration over, say, the s -variables in Eq. (1.3) is required, we have to deal with new types of boundary contributions which have no analogue in ordinary analysis. The general theory of this effect, which is sometimes overlooked in the literature, was given by Rothstein.¹⁵ In Refs. 7 and 16 it was shown how to treat these contributions in the case of the supersymmetric Itzykson-Zuber integral. Regarding integrations over the supersymmetric Itzykson-Zuber integral, yet another comment is in order: in applications in statistical mechanics,^{3,4,7,11,12} the integrand, containing formulae (1.1) or (1.3), possesses some invariances under permutations. This allows one to replace the numerator of the right hand sides of these equations by $\exp(i\text{tr}xy)$ or $\exp(i\text{tr}gsr)$, respectively, which makes the calculations more transparent. We emphasize this trivial point since it has stirred some confusion lately.

It is the purpose of this work to transfer all the results which have been reviewed so far from Hermitean to arbitrary complex matrices. We will do this for the case of ordinary and supermatrices. We will derive closed formulae for the analogues of the ordinary and the supersymmetric Itzykson-Zuber integral in the space of complex matrices. To the best of our knowledge, those have not been worked out yet. Our results are of considerable interest in the theory of matrix models. Several models in two-dimensional Quantum Gravity involve complex instead of Hermitean matrices.^{17,18} Recently, the so-called chiral Gaussian Ensembles,¹⁹ also based on complex matrices, have been introduced and proved to be very useful in studying certain aspects of Quantum Chromodynamics (QCD).^{20–23} In particular, the integral we compute will be crucial for a further analysis of the spectral correlations of the Dirac operator of QCD in the framework of random-matrix models at finite temperatures, in the presence of mass terms, or at finite chemical potential. Via the Banks-Casher formula,²⁴ such analysis is also important for the study of questions related to the chiral phase transition of QCD.

To derive our results, we transfer the diffusion equation technique to complex ordinary and supermatrices. Again, the crucial point turns out to be the separability in the curved space of, in this case, radial coordinates. We have no doubts that the explicit formula could also be evaluated using other techniques. Especially, Gelfand-Tzetlin coordinates could be used as in Ref. 6 for ordinary Hermitean matrices. Recently, Gelfand-Tzetlin coordinates were derived for Hermitean supermatrices¹⁴ such that this method could also be used for complex supermatrices. Moreover, regarding the case of ordinary matrices, we do not exclude the possibility that our results might be derivable directly from Harish-Chandra's formula.² To the best of our knowledge, Harish-Chandra's result has not been transferred to superanalysis yet. Nevertheless, here we will focus on the diffusion equation technique. So far, this technique was, for Hermitean matrices, viewed as a purely mathematical tool. However, as mentioned before, the diffusion in superspace describes the transition from arbitrary to chaotic fluctuations in Random Matrix Theory¹² and, therefore, also has a direct physical meaning. We strongly believe that similar features are likely to exist in the case of the diffusion in the space of complex matrices which we will discuss in the present work.

The paper is organized as follows. In Sec. II, we state our results and derive them by constructing an eigenvalue equation. In Sec. III, we discuss the diffusion and questions related to it. We summarize and discuss our results in Sec. IV. Three appendices are provided for the derivation of intermediate results used in the text. In a fourth appendix, we discuss some boundary contributions which occur in the case of Hermitean Supermatrices.

II. DERIVATION OF THE INTEGRAL BY CONSTRUCTING AN EIGENVALUE EQUATION

In Sec. II A, we state the integral for both cases, for ordinary and supermatrices. The derivation is performed for ordinary and supermatrices in Secs. II B and II C, respectively.

A. Statement of the Integral

Let X be an arbitrary, square, complex ordinary matrix of dimension N . It is well known²⁵ that it can be written in the so-called pseudo-diagonal form

$$X = U x \bar{V} \quad \text{with} \quad x = \text{diag}(x_1, \dots, x_N) , \quad (2.1)$$

where the N variables x_n are real and non-negative. Note that these are not eigenvalues, they will be referred to as radial coordinates. While the matrix U explores the full parameter space of the unitary group $U(N)$, the matrix \bar{V} is, in order to remove a double counting of phases, restricted to a subspace defined as the quotient of the unitary group and the Cartan subgroup, hence we have $U \in U(N)$ and $\bar{V} \in U(N)/U^N(1)$.

We now multiply X by a diagonal matrix y of the same form as x and consider the expression $i \text{Re} \text{tr} U x \bar{V} y$. The integral we wish to compute is the angular average over both unitary matrices of the exponential of this trace,

$$\Phi(x, y) = \int d\mu(U) \int d\mu(\bar{V}) \exp(i \text{Re} \text{tr} U x \bar{V} y) , \quad (2.2)$$

with $d\mu(U)$ and $d\mu(\bar{V})$ being the corresponding invariant Haar measures. We show that this integral is given by

$$\Phi(x, y) = \frac{(2\pi)^{N^2}}{N!} \frac{\det [J_0(x_n y_m)]_{n,m=1,\dots,N}}{\Delta_N(x^2) \Delta_N(y^2)} , \quad (2.3)$$

where $J_0(z)$ is the ordinary Bessel function of zeroth order. The Vandermonde determinant was defined in Eq. (1.2).

Remarkably and fully analogously to the case of Hermitean matrices, this result can straightforwardly be generalized to supermatrices. An arbitrary complex supermatrix σ of dimension $k_1 + k_2$ can be written as

$$\sigma = u s \bar{v} \quad \text{with} \quad s = \text{diag}(s_1, i s_2) \quad (2.4)$$

and $s_j = \text{diag}(s_{1j}, \dots, s_{kj})$ for $j = 1, 2$. Again, while the matrix u explores the full parameter space of the unitary supergroup, the matrix \bar{v} has to be restricted to a subspace in order to remove phases, we thus have $u \in U(k_1/k_2)$ and $\bar{v} \in U(k_1/k_2)/U^{k_1+k_2}(1)$.

Analogously to the case of ordinary matrices, we multiply the matrix σ by a diagonal matrix r of the same form as s and consider the expression $\text{Re} \text{tr} \sigma u s \bar{v} r$. The generalization of the integral in the ordinary case is given by replacing the trace by the supertrace and the invariant measures by the corresponding ones in superspace, $d\mu(u)$ and $d\mu(\bar{v})$, respectively. We show that the double average

$$\varphi(s, r) = \int d\mu(u) \int d\mu(\bar{v}) \exp(i \text{Re} \text{tr} \sigma u s \bar{v} r) \quad (2.5)$$

is given by

$$\varphi(s, r) = \frac{(2\pi)^{(k_1-k_2)^2}}{2^{2k_1 k_2} k_1! k_2!} \frac{\det [J_0(s_{p1} r_{p'1})]_{p,p'=1,\dots,k_1} \det [J_0(s_{q2} r_{q'2})]_{q,q'=1,\dots,k_2}}{B_{k_1 k_2}(s^2) B_{k_1 k_2}(r^2)} , \quad (2.6)$$

where the generalized Vandermonde determinant was defined in Eq. (1.4).

Obviously, formula (2.3) includes formula (2.6) as can be seen by putting $k_1 = N$ and $k_2 = 0$. Hence, in principle, it is sufficient to perform the derivation solely in superspace. However, we decided not to do so. We prove both results separately, first, in order to give those readers with little interest in supermathematics the opportunity to understand the ordinary case without being burdened by undesired information and, second, in order to help those readers with little experience in supermathematics to approach this area starting from more familiar grounds.

Note that our discussion is related to the harmonic analysis in the corresponding matrix spaces. The functions $\Phi(x, y)$ and $\varphi(s, r)$ can be viewed as the lowest order Bessel functions in these spaces.

B. Derivation for Ordinary Matrices

Besides X , we introduce a second arbitrary complex matrix Y of dimension N whose pseudo-diagonalization reads $Y = U'y\bar{V}'$ with y defined in Sec. II A. We have $U' \in \mathrm{U}(N)$ and $\bar{V}' \in \mathrm{U}(N)/\mathrm{U}^N(1)$. We observe that the “plane wave”

$$W(X, Y) = \exp(i \mathrm{Re} \mathrm{tr} XY^\dagger) \quad (2.7)$$

in this matrix space obeys the “wave equation”

$$\Delta W(X, Y) = -(\mathrm{tr} YY^\dagger) W(X, Y) , \quad (2.8)$$

where the Laplacian is defined as

$$\Delta = \sum_{n,m} \left(\frac{\partial^2}{\partial(\mathrm{Re} X_{nm})^2} + \frac{\partial^2}{\partial(\mathrm{Im} X_{nm})^2} \right) . \quad (2.9)$$

Due to the invariance of the Haar measures we can express the function (2.2) as the angular average of the matrix plane wave,

$$\Phi(x, y) = \int d\mu(U') \int d\mu(\bar{V}') W(X, Y) . \quad (2.10)$$

The crucial observation is now, just as in the case of Hermitean matrices, that $\Phi(x, y)$ satisfies the wave equation (2.8). This can be seen by averaging both sides of (2.8) over the matrices U' and \bar{V}' using $\mathrm{tr} YY^\dagger = \mathrm{tr} y^2$. Consequently, since $\Phi(x, y)$ depends only on the radial coordinates, we can replace the Laplacian Δ by its radial part Δ_x . To construct it we have to transform the Cartesian volume element

$$d[X] = \prod_{n,m} d\mathrm{Re} X_{nm} d\mathrm{Im} X_{nm} \quad (2.11)$$

to radial and angular coordinates,

$$\begin{aligned} d[X] &= J(x) d[x] d\mu(U) d\mu(\bar{V}) \\ d[x] &= \prod_{n=1}^N dx_n \\ J(x) &= \Delta_N^2(x^2) \prod_{n=1}^N x_n , \end{aligned} \quad (2.12)$$

where the Jacobian $J(x)$ was worked out in Ref. 25. It is then easily shown that the radial part of the Laplace operator reads

$$\Delta_x = \sum_{n=1}^N \frac{1}{J(x)} \frac{\partial}{\partial x_n} J(x) \frac{\partial}{\partial x_n} , \quad (2.13)$$

and we thus arrive at the eigenvalue equation

$$\Delta_x \Phi(x, y) = -(\mathrm{tr} y^2) \Phi(x, y) \quad (2.14)$$

in the curved space of the radial coordinates.

The key to the solution of the above equation is the separability of the radial Laplacian. For an arbitrary function $\Lambda(x)$ we have the identity

$$\Delta_x \frac{\Lambda(x)}{\Delta_N(x^2)} = \frac{1}{\Delta_N(x^2)} \Delta'_x \Lambda(x) , \quad (2.15)$$

where the reduced part of the radial Laplacian is

$$\Delta'_x = \sum_{n=1}^N \left(\frac{\partial^2}{\partial x_n^2} + \frac{1}{x_n} \frac{\partial}{\partial x_n} \right) . \quad (2.16)$$

The proof of this fact is given in Appendix A. Hence, the ansatz

$$\Phi(x, y) = \frac{\Psi(x, y)}{\Delta_N(x^2) \Delta_N(y^2)} \quad (2.17)$$

in which, for symmetry reasons, x and y are treated on the same footing, reduces the radial equation (2.14) to the much simpler form

$$\Delta'_x \Psi(x, y) = -(\text{tr } y^2) \Psi(x, y) . \quad (2.18)$$

This equation is again separable by a product ansatz for $\Psi(x, y)$ which yields N Bessel differential equations of zeroth order for each of the N functions. Hence, each of them can be written as a linear combination of the Bessel and Weber functions J_0 and N_0 , respectively. Note that the indices of the eigenvalues y_n^2 of these N Bessel differential equations can be permuted arbitrarily implying that the most general solution is a linear combination of all these permuted products. However, the integral representation (2.2) imposes certain boundary conditions on the solution of the differential equation (2.18). Since the integral has a finite value for all x and y we have to exclude the Weber function from the solution. Additionally, we have to take into account that the integral is invariant under permutations of the indices. Since $\Delta_N(x^2)$, the Vandermonde determinant, is really a determinant, the function $\Psi(x, y)$ has to have the same properties under permutations of the indices. Incorporating these boundary conditions we find

$$\Psi(x, y) = \frac{(2\pi)^{N^2}}{N!} \det [J_0(x_n y_m)]_{n,m=1,\dots,N} , \quad (2.19)$$

which yields immediately the result (2.3). Of course, the normalization constant is arbitrary. We will show later why our choice is useful.

C. Derivation for Supermatrices

All steps are completely analogous to the ordinary case. In order to make the notation more transparent, we write the supermatrix σ in the boson-fermion block form^{7,10}

$$\sigma = \begin{bmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{21} & i\sigma^{22} \end{bmatrix} , \quad (2.20)$$

where σ^{jl} is a $k_j \times k_l$ complex matrix whose entries are commuting if $j = l$ and anticommuting if $j \neq l$. The factor i in front of σ^{22} is, as usual, introduced to ensure convergence.^{9,10} Again, besides σ , we introduce a second arbitrary complex supermatrix ρ of the same form whose pseudo-diagonalization reads $\rho = u' r \bar{v}'$ with r defined in Sec. II A and with $u' \in \text{U}(k_1/k_2)$ and $\bar{v}' \in \text{U}(k_1/k_2)/\text{U}^{k_1+k_2}(1)$. There is also a “plane wave”

$$w(\sigma, \rho) = \exp(i \text{Re} \text{trg} \sigma \rho^\dagger) \quad (2.21)$$

in this matrix space. Note that the expression $\text{Re} \text{trg} \sigma \rho^\dagger$ has, for explicit calculations, to be interpreted as half the sum of $\text{trg} \sigma \rho^\dagger$ and its complex conjugate since we will not introduce the real and the imaginary part of Grassmann variables. The plane wave satisfies the “wave equation”

$$\Delta w(\sigma, \rho) = -(\text{trg} \rho \rho^\dagger) w(\sigma, \rho) , \quad (2.22)$$

where the Laplacian is defined as

$$\Delta = \sum_{j=1}^2 \sum_{p,q} \left(\frac{\partial^2}{\partial (\text{Re } \sigma_{pq}^{jj})^2} + \frac{\partial^2}{\partial (\text{Im } \sigma_{pq}^{jj})^2} \right) + 4 \sum_{j \neq l} \sum_{p,q} \frac{\partial^2}{\partial \sigma_{pq}^{jl} \sigma_{pq}^{jl*}} . \quad (2.23)$$

The invariance of the Haar measures allows us to express the function (2.5) as the angular average of the matrix plane wave,

$$\varphi(s, r) = \int d\mu(u') \int d\mu(\bar{v}') w(\sigma, \rho) . \quad (2.24)$$

As in the ordinary case, we integrate both sides of the wave equation over the matrices u' and \bar{v}' using $\text{tr} g \rho \rho^\dagger = \text{tr} g r^2$ and observe that $\varphi(s, r)$ satisfies the wave equation (2.22). Again, since $\varphi(s, r)$ depends only on the radial coordinates, we can replace the Laplacian Δ by its radial part Δ_s . The transformation of the Cartesian volume element

$$d[\sigma] = \prod_{j=1}^2 \prod_{p,q} d(\text{Re } \sigma_{pq}^{jj}) d(\text{Im } \sigma_{pq}^{jj}) \prod_{j \neq l} \prod_{p,q} d\sigma_{pq}^{jl*} d\sigma_{pq}^{jl} \quad (2.25)$$

to radial and angular coordinates reads

$$\begin{aligned} d[\sigma] &= J(s) d[s] d\mu(u) d\mu(\bar{v}) \\ d[s] &= \prod_{j=1}^2 \prod_{p=1}^{k_j} ds_{pj} \\ J(s) &= B_{k_1 k_2}^2(s^2) \prod_{j=1}^2 \prod_{p=1}^{k_j} s_{pj} , \end{aligned} \quad (2.26)$$

where the Jacobian or Berezinian $J(s)$ is computed in Appendix B. The radial part of the Laplace operator takes the form

$$\Delta_s = \sum_{j=1}^2 \sum_{p=1}^{k_j} \frac{1}{J(s)} \frac{\partial}{\partial s_{pj}} J(s) \frac{\partial}{\partial s_{pj}} , \quad (2.27)$$

details are given in Appendix B. Hence, we have to solve the equation

$$\Delta_s \varphi(s, r) = -(\text{tr} g r^2) \varphi(s, r) \quad (2.28)$$

in the curved space of the radial coordinates.

In the case of ordinary matrices, the key for the solution was the separability of the radial Laplacian. It is essential that this feature is also present in the case of supermatrices. This closely parallels the situation for Hermitean matrices.⁷ For an arbitrary function $\lambda(s)$ we find

$$\Delta_s \frac{\lambda(s)}{B_{k_1 k_2}(s^2)} = \frac{1}{B_{k_1 k_2}(s^2)} \Delta'_s \lambda(s) , \quad (2.29)$$

where the reduced part of the Laplacian reads

$$\Delta'_s = \sum_{j=1}^2 \sum_{p=1}^{k_j} \left(\frac{\partial^2}{\partial s_{pj}^2} + \frac{1}{s_{pj}} \frac{\partial}{\partial s_{pj}} \right) . \quad (2.30)$$

The derivation is given in Appendix C. Thus, the ansatz

$$\varphi(s, r) = \frac{\psi(s, r)}{B_{k_1 k_2}(s^2) B_{k_1 k_2}(r^2)} \quad (2.31)$$

yields the reduced equation

$$\Delta'_s \psi(s, r) = -(\text{tr} g r^2) \psi(s, r) \quad (2.32)$$

which, again, is separable by a product ansatz for $\psi(s, r)$. We obtain $k_1 + k_2$ Bessel differential equations of zeroth order. The boundary conditions imposed by the integral representation (2.5) are very similar to the ones in the ordinary case. First, we have to construct the solution using the Bessel function J_0 and to reject the Weber function N_0 . Second, we have to take into account the invariance under permutations. Here, however, we see from the Jacobian that this invariance exists only within the boson-boson or fermion-fermion block, respectively. Since these boundary conditions imply that the solution is given by

$$\psi(s, r) = \frac{(2\pi)^{(k_1 - k_2)^2}}{2^{2k_1 k_2} k_1! k_2!} \det [J_0(s_{p1} r_{p'1})]_{p,p'=1,\dots,k_1} \det [J_0(s_{q2} r_{q'2})]_{q,q'=1,\dots,k_2} , \quad (2.33)$$

we arrive at the final result (2.6). Again, the normalization constant is arbitrary, we will comment on our choice later.

III. DIFFUSION EQUATION AND FOURIER TRANSFORM

We now discuss a diffusion equation which is closely related to the plane waves and the eigenvalue equations we constructed in the previous section. Our goal is to show that the concept of diffusion equations which is so useful in the case of Hermitean matrices can be transferred straightforwardly to arbitrary complex matrices. However, since these considerations are more of conceptual interest and do not require so many explicit calculations, we study only the case of supermatrices. The case of ordinary matrices is always recovered by setting $k_1 = N$ and $k_2 = 0$. In Sec. III A, we introduce the concepts in Cartesian space. We go over to the curved space of the radial coordinates in Sec. III B. In Sec. III C, we discuss some questions related to boundary terms.

A. Cartesian Space

We introduce a time coordinate t and consider the diffusion equation

$$\frac{1}{2}\Delta F(\sigma, t) = \frac{\partial}{\partial t} F(\sigma, t) \quad (3.1)$$

for a given initial condition $F_0(\sigma)$ such that

$$\lim_{t \rightarrow 0} F(\sigma, t) = F_0(\sigma) . \quad (3.2)$$

The kernel of this diffusion satisfies the equations

$$\frac{1}{2}\Delta G(\sigma, t) = \frac{\partial}{\partial t} G(\sigma, t) \quad \text{and} \quad \lim_{t \rightarrow 0} G(\sigma, t) = \delta(\sigma) , \quad (3.3)$$

where the δ -function is given by

$$\delta(\sigma) = \prod_{j=1}^2 \prod_{p,q} \delta(\text{Re } \sigma_{pq}^{jj}) \delta(\text{Im } \sigma_{pq}^{jj}) \prod_{j \neq l} \prod_{p,q} \delta(\sigma_{pq}^{jl*}) \delta(\sigma_{pq}^{jl}) . \quad (3.4)$$

The δ -function of an anticommuting variable β is defined by $\delta(\beta) = \sqrt{2\pi}\beta$.^{7,10} Similar to the discussion in Refs. 1 and 16, the kernel is the Gaussian

$$G(\sigma, t) = \frac{2^{2k_1 k_2}}{(2\pi t)^{(k_1 - k_2)^2}} \exp\left(-\frac{1}{2t} \text{trg} \sigma \sigma^\dagger\right) , \quad (3.5)$$

and the solution of the diffusion process can be written as the convolution

$$F(\sigma, t) = \int G(\sigma - \sigma', t) F_0(\sigma') d[\sigma'] . \quad (3.6)$$

Moreover, this solution is also expressible as

$$F(\sigma, t) = \exp\left(\frac{t}{2}\Delta\right) F_0(\sigma) \quad (3.7)$$

which has to be viewed as a formal power series.

We will now show how the diffusion can be related to the plane waves of the previous section and to the theory of Fourier transforms. To this end, we remark that the δ -function (3.4) can be expanded in the plane waves (2.21),

$$\delta(\sigma) = \frac{2^{4k_1 k_2}}{(2\pi)^{2(k_1 - k_2)^2}} \int w(\sigma, \rho) d[\rho] , \quad (3.8)$$

which allows us to introduce the Fourier transform $\tilde{P}(\rho)$ of a function $P(\sigma)$ and its inverse by

$$\tilde{P}(\rho) = \frac{2^{2k_1 k_2}}{(2\pi)^{(k_1 - k_2)^2}} \int P(\sigma) w^*(\sigma, \rho) d[\sigma] \quad \text{and} \quad P(\sigma) = \frac{2^{2k_1 k_2}}{(2\pi)^{(k_1 - k_2)^2}} \int \tilde{P}(\rho) w(\sigma, \rho) d[\rho] . \quad (3.9)$$

The Fourier transform can be used to derive the explicit form (3.5) of the diffusion kernel defined in Eq. (3.3), this works as follows. The diffusion kernel can, according to Eq. (3.7), be expressed in the form

$$G(\sigma, t) = \exp\left(\frac{t}{2}\Delta\right)\delta(\sigma) \quad (3.10)$$

in which we insert the expansion (3.8),

$$G(\sigma, t) = \frac{2^{4k_1 k_2}}{(2\pi)^{2(k_1 - k_2)^2}} \int \exp\left(\frac{t}{2}\Delta\right) w(\sigma, \rho) d[\rho] . \quad (3.11)$$

We write the exponential as a power series and, by virtue of the eigenvalue equation (2.22), perform all derivatives. The resummation of the series gives the diffusion kernel as the Fourier transform of a Gaussian

$$G(\sigma, t) = \frac{2^{4k_1 k_2}}{(2\pi)^{2(k_1 - k_2)^2}} \int \exp\left(-\frac{t}{2}\text{tr}g\rho\rho^\dagger\right) w(\sigma, \rho) d[\rho] , \quad (3.12)$$

which is in agreement with Eq. (3.5).

B. Curved Space of the Radial Coordinates

We now assume that the initial condition depends only on the radial coordinates, i.e. $F_0(\sigma) = F_0(s)$. Thus, it is useful to use the coordinates (2.4) in the integral representation (3.6) of the solution of the diffusion equation (3.1). This has some important consequences. The invariance of the Haar measures implies that this solution is also a function of the radial coordinates only, hence we have $F(\sigma, t) = F(s, t)$. Consequently, the diffusion takes place in the curved space of the radial coordinates alone,

$$\frac{1}{2}\Delta_s F(s, t) = \frac{\partial}{\partial t} F(s, t) \quad \text{and} \quad \lim_{t \rightarrow 0} F(s, t) = F_0(s) , \quad (3.13)$$

where Δ_s is the radial part of the Laplacian defined in Eq. (2.27). Moreover, we may conclude from the integral representation (3.6) that the kernel of the diffusion (3.13) is given by

$$\Gamma(s, s', t) = \int d\mu(u) \int d\mu(\bar{v}) G(us\bar{v} - s', t) . \quad (3.14)$$

There are two ways of evaluating this double average. First, since the kernel $G(\sigma, t)$ is Gaussian, a direct comparison of Eq. (3.5) to Eq. (2.5) shows that

$$\Gamma(s, s', t) = \frac{2^{2k_1 k_2}}{(2\pi t)^{(k_1 - k_2)^2}} \exp\left(-\frac{1}{2t}\text{tr}g(s^2 + s'^2)\right) \varphi(-is/t, s') , \quad (3.15)$$

which means that this double average can be expressed in terms of the one we have calculated in the previous section. Hence, with the help of the result (2.6) and after a reordering of factors, we can write the diffusion kernel in the curved space in the form

$$\Gamma(s, s', t) = \frac{1}{k_1! k_2!} \frac{\det[\gamma(s_{p1}, s'_{p'1}, t)]_{p, p'=1, \dots, k_1} \det[\gamma(s_{q2}, s'_{q'2}, t)]_{q, q'=1, \dots, k_2}}{B_{k_1 k_2}(s^2) B_{k_1 k_2}(s'^2)} , \quad (3.16)$$

in which the entries of the determinants are given by the function

$$\gamma(s_{pj}, s'_{qj}, t) = \frac{1}{t} \exp\left(-\frac{s_{pj}^2 + s'_{qj}^2}{2t}\right) I_0\left(\frac{s_{pj} s'_{qj}}{t}\right) \quad (3.17)$$

for all values of $j = 1, 2$ and $p, q = 1, \dots, k_j$. The function $I_0(z)$ is the modified Bessel function of zeroth order.

Alternatively, if the result (2.6) was unknown, formula (3.16) could be derived by a procedure similar to the one in Sec. II C. The separability of the radial part Δ_s leads to a reduced diffusion equation involving the reduced operator Δ'_s defined in (2.30). This equation can be solved by a product ansatz leading to the diffusion equation

$$\frac{1}{2} \left(\frac{\partial^2}{\partial s_{pj}^2} + \frac{1}{s_{pj}} \frac{\partial}{\partial s_{pj}} \right) \gamma(s_{pj}, s'_{qj}, t) = \frac{\partial}{\partial t} \gamma(s_{pj}, s'_{qj}, t) , \quad (3.18)$$

where the differential operator is just the radial part of the Laplacian in a two-dimensional space. In order to construct the solution, we express it as the formal series

$$\gamma(s_{pj}, s'_{qj}, t) = \exp \left(\frac{t}{2} \left(\frac{\partial^2}{\partial s_{pj}^2} + \frac{1}{s_{pj}} \frac{\partial}{\partial s_{pj}} \right) \right) \frac{\delta(s_{pj} - s'_{qj})}{\sqrt{s_{pj} s'_{qj}}} \quad (3.19)$$

acting on the proper radial δ -function in this two-dimensional space. Inserting Hankel's expansion²⁶ of this δ -function,

$$\frac{\delta(s_{pj} - s'_{qj})}{\sqrt{s_{pj} s'_{qj}}} = \int_0^\infty J_0(s_{pj} z) J_0(s'_{qj} z) z dz , \quad (3.20)$$

we can perform all derivatives and resum the series. We arrive at

$$\gamma(s_{pj}, s'_{qj}, t) = \int_0^\infty \exp \left(-\frac{t}{2} z^2 \right) J_0(s_{pj} z) J_0(s'_{qj} z) z dz , \quad (3.21)$$

which is precisely Weber's representation²⁶ of the function (3.17). It is easy to see in a direct calculation that this function is indeed the kernel of the diffusion equation (3.18). According to an elementary result²⁶ of the theory of Bessel functions, $\gamma(s_{pj}, s'_{qj}, t)$ is properly normalized,

$$\int_0^\infty \gamma(s_{pj}, s'_{qj}, t) s_{pj} ds_{pj} = 1 , \quad (3.22)$$

where we have used the radial part $s_{pj} ds_{pj}$ of the measure in the two-dimensional space. Furthermore, since $I_0(z)$ behaves like $\exp(z)/\sqrt{2\pi z}$ for large values of the argument, the function $\gamma(s_{pj}, r_{qj}, t)$ approaches the δ -function

$$\lim_{t \rightarrow 0} \gamma(s_{pj}, s'_{qj}, t) = \frac{\delta(s_{pj} - s'_{qj})}{\sqrt{s_{pj} s'_{qj}}} \quad (3.23)$$

for vanishing time t .

The limit relation (3.23) implies that the kernel (3.16) satisfies the correct limit relation in the curved space of all radial coordinates, we write this in the form

$$\begin{aligned} \lim_{t \rightarrow 0} \Gamma(s, s', t) &= \int d\mu(u) \int d\mu(\bar{v}) \delta(us\bar{v} - s') \\ &= \frac{1}{k_1! k_2!} \frac{\det [\delta(s_{p1} - s'_{p'1})]_{p,p'=1,\dots,k_1} \det [\delta(s_{q2} - s'_{q'2})]_{q,q'=1,\dots,k_2}}{\sqrt{J(s) J(s')}} , \end{aligned} \quad (3.24)$$

where the Berezinian $J(s)$ is defined in Eq. (2.26). Using this result, it is easily checked that the constant $1/k_1! k_2!$ ensures the correct normalization. This, in turn, motivates our choice of the normalization constants in Eqs. (2.19) and (2.33).

C. Questions Related to Boundary Contributions

The function $\Gamma(s, s', t)$ given in Eq. (3.16) is, as we have shown, the kernel of the diffusion equation (3.13) in the curved space of the radial coordinates. Thus, the solution of the integral (3.14) as it stands is indeed given by formula (3.16). However, there is a very subtle point about kernels of this type which has an important impact on applications. Although we will present some applications of our results to physical problems in a forthcoming

publication, we already give a short, more intuitive, discussion of this subtlety here. We do so to acquaint the reader who is not yet familiar with supersymmetry with this point which is sometimes overlooked in the literature.

In Cartesian space, the normalization of the Gaussian diffusion kernel (3.5) implies that the equation

$$\int G(\sigma - \sigma', t) d[\sigma'] = 1 \quad (3.25)$$

holds for all values of t and for all matrices σ . Thus, after performing the angular integration, one would naively assume that the radial integral

$$\eta(s, t) = \int \Gamma(s, s', t) J(s') d[s'] \quad (3.26)$$

also yields unity for all values of t and for all diagonal matrices s . In the ordinary case, i.e. for $k_1 = N$ and $k_2 = 0$, it can be checked by a straightforward, direct calculation that we indeed have $\eta(s, t) = 1$. However, in the case of supermatrices this is, for non-trivial reasons, no longer true. The singularities of the Berezinian $J(s)$ compensate the vanishing of some angular Grassmann integrals such that a finite, non-zero result remains. This effect leads to certain contributions to the integral which are often called Efetov-Wegner-Parisi-Sourlas terms in the more physics-oriented literature. There are various methods to construct those contributions in the applications of supersymmetry.^{7,9–11,27} From a strictly mathematical point of view, these terms arise as boundary contributions due to the fact that the integrals are ill-defined for non-compact supermanifolds. A full-fledged theory can be found in Ref. 15. It is instructive to think of these boundary contributions as being necessary to restore the translational invariance of the integrals in Cartesian space, as obvious in Eq. (3.25), which is broken in Eq. (3.26) if $\eta(s, t)$ differs from unity.²⁸ To illustrate this, we calculate the function $\eta(s, t)$ for the simplest non-trivial case, namely $k_1 = k_2 = 1$. We have

$$\eta(s, t) = (s_{11}^2 + s_{12}^2) \int_0^\infty \int_0^\infty ds'_{11} ds'_{12} \frac{s'_{11} s'_{12}}{s'^2_{11} + s'^2_{12}} \gamma(s_{11}, s'_{11}, t) \gamma(s_{12}, s'_{12}, t) . \quad (3.27)$$

By expressing the denominator of the Berezinian as the integral

$$\frac{1}{s'^2_{11} + s'^2_{12}} = \int_0^\infty \exp(-(s'^2_{11} + s'^2_{12})z) dz , \quad (3.28)$$

we can evaluate the double integral (3.27) by standard methods.²⁶ We arrive at

$$\eta(s, t) = 1 - \exp\left(-\frac{s_{11}^2 + s_{12}^2}{2t}\right) \quad (3.29)$$

which equals unity only in the limit $t \rightarrow 0$. Note that the exponential is, apart from a numerical factor, nothing else but the Cartesian kernel at $\sigma' = 0$ which is just $G(\sigma, t) = G(s, t)$. This is, of course, no accidental coincidence.

The case $k_1 = k_2$, where k_1 is arbitrary, is physically the most interesting one. Due to the determinant structure of the function $B_{k_1 k_1}(s'^2)$, the evaluation of the function $\eta(s, t)$ reduces to the double integral (3.27) already computed, and we arrive at

$$\eta(s, t) = \frac{1}{B_{k_1 k_1}(s^2)} \det \left[\frac{1}{s_{p1}^2 + s_{q2}^2} \left(1 - \exp\left(-\frac{s_{p1}^2 + s_{q2}^2}{2t}\right) \right) \right]_{p,q=1,\dots,k_1} . \quad (3.30)$$

As evident from the definition (3.26), this function is a solution of diffusion equation (3.13), we have

$$\frac{1}{2} \Delta_s \eta(s, t) = \frac{\partial}{\partial t} \eta(s, t) . \quad (3.31)$$

However, it is not a kernel of the diffusion process in the usual sense since it obeys different limit relations,

$$\lim_{\substack{t \rightarrow 0 \\ s \neq 0}} \eta(s, t) = 1 \quad \text{and} \quad \lim_{t \rightarrow \infty} \eta(s, t) = 0 , \quad (3.32)$$

which reflect the existence of the new boundary contributions. The function $\eta(s, t)$ can be viewed as the envelope solution corresponding to the kernel $\Gamma(s, s', t)$. The function $B_{k_1 k_1}(s^2) \Gamma(s, s', t)$ possesses a product structure in the

kernels $\gamma(s_{p1}, s'_{q2}, t)$. Due to the integration, this property has vanished in $B_{k_1 k_1}(s^2)\eta(s, t)$, which factorizes only in functions of the combinations $s_{p1}^2 + s_{q2}^2$. Along the lines given in Refs. 7, 10, and 28, one can show that the diffusion kernel in the curved space of the radial coordinates has to be replaced by

$$\Gamma(s, s', t) \longrightarrow (1 - \eta(s, t)) \frac{\delta(s')}{J(s')} + \Gamma(s, s', t) \quad (3.33)$$

if further integration over the primed variables s' is required. This replacement cures the problem of the boundary contributions for $k_1 = k_2$ in the physically most interesting cases. Hence, for a well-behaved initial condition $F_0(s)$, the solution $F(s, t)$ of the diffusion equation in the curved space of the radial coordinates reads

$$F(s, t) = (1 - \eta(s, t))F_0(0) + \int \Gamma(s, s', t) F_0(s') J(s') d[s'] . \quad (3.34)$$

We emphasize that this result is really a solution of the diffusion process (3.13), including its initial condition. Note that there are very peculiar cases in which further boundary contributions can arise. Those, however, have to be constructed using the full theory which is given in Ref. 15.

In Appendix D, we reconsider the boundary contributions to the supersymmetric Itzykson-Zuber integral for Hermitean Supermatrices as derived in Ref. 7.

IV. SUMMARY AND DISCUSSION

We have calculated an analogue of the Itzykson-Zuber integral in the space of arbitrary complex matrices. We arrived at explicit formulae for the case of ordinary and supermatrices, where the latter includes the former. We performed our calculation by transferring the diffusion equation technique of Itzykson and Zuber for Hermitian matrices, which works in ordinary¹ and in superspace,^{7,13} to complex matrices. For the actual derivation, we used an eigenvalue equation for the plane waves in these matrix spaces which is closely related to this diffusion. Similar to the Hermitean case, the integral in question turns out to be the kernel of the diffusion in the curved spaces of the radial coordinates. The explicit results can be computed due to a separability of the Laplacian in these radial spaces. We discussed certain types of boundary contributions to the full solution of the diffusion equation which can arise in superspace.

We have no doubts that our explicit results can also be derived by other methods. In particular, the use of Gelfand-Tzetlin coordinates for the unitary group in ordinary⁶ and superspace¹⁴ ought to be mentioned here since it allows a recursive evaluation of correlation functions in the corresponding matrix models. Most importantly, as far as the case of ordinary matrices is concerned, it does not seem unlikely that the explicit formula for the integral we presented here can be derived directly from the Harish-Chandra integral. At first sight, one would not think so since the Harish-Chandra integral is an average over one unitary matrix whereas our result is a double average over two unitary matrices. This can be seen from the fact that our explicit formula contains Bessel functions where the Itzykson-Zuber integrals contain exponentials, i.e. plane waves. The Bessel function of zeroth order is just the angular average over a plane wave in a two-dimensional space. This might indicate that the angular average over two unitary matrices is essential and can not be replaced by an average over one unitary matrix. Nevertheless, we do not exclude the possibility that a clever reordering of the trace in the matrix plane waves which combines these two unitary matrices can be done in such a way that the essential part of the calculation reduces to an application of the Harish-Chandra formula. These considerations, however, do not apply to the case of supermatrices, for which, to the best of our knowledge, Harish-Chandra's result has not been transferred yet.

For the reasons discussed above, we do not want to present our explicit formulae for the integrals as our most important findings. Rather, we consider the Itzykson-Zuber-like diffusion which we constructed here as our most interesting result. We strongly believe that this diffusion is more than a mathematical tool to calculate integrals. In the case of Hermitean supermatrices, it was shown¹² that the Itzykson-Zuber diffusion models the transition from arbitrary to chaotic fluctuations of all orders in a very general way. We are convinced that the diffusion in the space of complex matrices also has a physical meaning of similar significance.

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APPENDIX A: SEPARABILITY IN ORDINARY SPACE

We use a more convenient form of Δ_x ,

$$\Delta_x = \sum_{n=1}^N \left(\frac{\partial^2}{\partial x_n^2} + \frac{\partial \ln J(x)}{\partial x_n} \frac{\partial}{\partial x_n} \right) . \quad (\text{A1})$$

The derivatives in Eq. (2.15) are evaluated in a straightforward manner, and we arrive at the intermediate result

$$\Delta_x \frac{\Lambda(x)}{\Delta_N(x^2)} = \frac{1}{\Delta_N(x^2)} \sum_{n=1}^N \left(\frac{\partial^2}{\partial x_n^2} + \frac{1}{x_n} \frac{\partial}{\partial x_n} - 4D_n \right) \Lambda(x) , \quad (\text{A2})$$

where $D_n = S_n + x_n^2(S_n^2 - T_n)$ with

$$S_n = \sum_{\substack{m=1 \\ m \neq n}}^N \frac{1}{x_n^2 - x_m^2} \quad \text{and} \quad T_n = \sum_{\substack{m=1 \\ m \neq n}}^N \frac{1}{(x_n^2 - x_m^2)^2} . \quad (\text{A3})$$

We now show that $\sum_{n=1}^N D_n = 0$. The fact that

$$\sum_{n=1}^N S_n = \sum_{m \neq n} \frac{1}{x_n^2 - x_m^2} = 0 \quad (\text{A4})$$

is easily seen by renaming summation indices. The remaining term is

$$R = \sum_{n=1}^N x_n^2(S_n^2 - T_n) = \sum_{\Omega(n,m,m')} \frac{x_n^2}{(x_n^2 - x_m^2)(x_n^2 - x_{m'}^2)} , \quad (\text{A5})$$

where the symbol $\Omega(n,m,m')$ denotes summation over three indices n, m, m' which are pairwise different. We now rename $n \leftrightarrow m$ and $n \leftrightarrow m'$ to obtain

$$2R = \sum_{\Omega(n,m,m')} \left[\frac{x_m^2}{(x_m^2 - x_n^2)(x_m^2 - x_{m'}^2)} + \frac{x_{m'}^2}{(x_{m'}^2 - x_n^2)(x_{m'}^2 - x_m^2)} \right] = -R , \quad (\text{A6})$$

from which $R = 0$ follows immediately. This completes the proof.

APPENDIX B: DERIVATION OF THE BEREZINIAN AND THE RADIAL PART OF THE LAPLACIAN

We wish to compute the Berezinian of the transformation $\sigma = us\bar{v}$, defined in Eq. (2.4). We first construct

$$d\sigma = u(\bar{u}^{\dagger}dus + ds + s\bar{v}\bar{v}^{\dagger})\bar{v} . \quad (\text{B1})$$

Writing $u^{\dagger}du = du'$ and $d\bar{v}\bar{v}^{\dagger} = d\bar{v}'$, and noting that $s^{\dagger} = s$, we obtain the invariant length element

$$\text{trg } d\sigma d\sigma^{\dagger} = \text{trg } (du's + ds + s\bar{v}') (sdu'^{\dagger} + ds + d\bar{v}'^{\dagger}s) \quad (\text{B2})$$

from which the Berezinian can be read off. Since du' and $d\bar{v}'$ are also in the algebra we are entitled to drop the primes. This gives

$$\text{trg } d\sigma d\sigma^{\dagger} = \text{trg } ds^2 + \text{trg } sds (du + du^{\dagger} + d\bar{v} + d\bar{v}^{\dagger}) + \text{trg } (dus + s\bar{v}) (sdu^{\dagger} + d\bar{v}^{\dagger}s) , \quad (\text{B3})$$

where we have made use of the fact that s and ds are diagonal. Since du and $d\bar{v}$ are anti-hermitian, the second term in the above expression yields zero. The vanishing of this term also shows that the Laplace operator separates into two sums over radial and angular coordinates, respectively, a fact which has been used in Sec. II C. The first term in Eq. (B3) contributes a factor of one to the Berezinian so that we are left with the third term only. Using boson-fermion block notation, we write

$$du = \begin{bmatrix} du^{C_1} & du^A \\ -du^{A^\dagger} & du^{C_2} \end{bmatrix} . \quad (\text{B4})$$

A note about the number of degrees of freedom: du^{C_j} is an anti-hermitian matrix with k_j^2 commuting degrees of freedom whereas du^A and du^{A^\dagger} each have $k_1 k_2$ anticommuting degrees of freedom. Similar notation is used for $d\bar{v}$, the main difference being that the diagonal elements of $d\bar{v}^{C_j}$ are zero. It is convenient to separate the diagonal elements of du^{C_j} and to define

$$dus + sd\bar{v} = \eta + \omega = \begin{bmatrix} \eta^{11} & 0 \\ 0 & \eta^{22} \end{bmatrix} + \begin{bmatrix} \omega^{11} & \omega^{12} \\ \omega^{21} & \omega^{22} \end{bmatrix} , \quad (\text{B5})$$

where $\eta^{11} = \text{diag}(du_{11}^{C_1} s_{11}, \dots, du_{k_1 k_1}^{C_1} s_{k_1 k_1})$, $\eta^{22} = \text{diag}(du_{11}^{C_2} i s_{12}, \dots, du_{k_2 k_2}^{C_2} i s_{k_2 k_2})$, the diagonal elements of ω^{11} and ω^{22} are zero, and

$$\begin{aligned} \omega_{pp'}^{11} &= d\bar{u}_{pp'}^{C_1} s_{p'1} + s_{p1} d\bar{v}_{pp'}^{C_1} & (p \neq p') \\ \omega_{qq'}^{22} &= d\bar{u}_{qq'}^{C_2} i s_{q'2} + i s_{q2} d\bar{v}_{qq'}^{C_2} & (q \neq q') \\ \omega_{pq}^{12} &= d\bar{u}_{pq}^A i s_{q2} + s_{p1} d\bar{v}_{pq}^A \\ \omega_{qp}^{21} &= -d\bar{u}_{qp}^{A^\dagger} s_{p1} - i s_{q2} d\bar{v}_{qp}^{A^\dagger} . \end{aligned} \quad (\text{B6})$$

We now consider

$$\begin{aligned} \text{trg} (\eta + \omega) (\eta^\dagger + \omega^\dagger) &= \text{trg} (\eta \eta^\dagger + \omega \omega^\dagger) \\ &= \text{trg} \eta \eta^\dagger + \text{tr} \omega^{11} \omega^{11\dagger} - \text{tr} \omega^{22} \omega^{22\dagger} - \text{tr} (\omega^{12} \omega^{12\dagger} + \omega^{21} \omega^{21\dagger}) , \end{aligned} \quad (\text{B7})$$

where in the first equality we have employed the fact that η is diagonal and that the diagonal elements of ω are zero by definition. Each independent variable appears in one and only one of the four terms in the above expression so that their contribution to the Berezinian is multiplicative. The contribution from the first term can be read off immediately, we obtain

$$\text{trg} \eta \eta^\dagger \rightarrow \prod_{j=1}^2 \prod_{p=1}^{k_j} s_{pj} . \quad (\text{B8})$$

We now write $\text{tr} \omega^{11} \omega^{11\dagger} = \sum_{p < p'} (|\omega_{pp'}^{11}|^2 + |\omega_{p'p}^{11}|^2)$. Each term in the sum contains only independent variables which do not appear in any other term so that the contribution of these terms to the Berezinian is multiplicative again. Using Eq. (B6) and the anti-hermiticity of $d\bar{u}^{C_1}$ and $d\bar{v}^{C_1}$ we obtain for the contribution to the Berezinian

$$\text{tr} \omega^{11} \omega^{11\dagger} \rightarrow \prod_{p < p'}^{k_1} (s_{p1}^2 - s_{p'1}^2)^2 = \Delta_{k_1}^2(s_1^2) . \quad (\text{B9})$$

In complete analogy,

$$-\text{tr} \omega^{22} \omega^{22\dagger} \rightarrow \prod_{q < q'}^{k_2} (s_{q2}^2 - s_{q'2}^2)^2 = \Delta_{k_2}^2(s_s^2) . \quad (\text{B10})$$

Similar arguments are made for the remaining term in Eq. (B7). Since ω^{12} and ω^{21} couple commuting and anticommuting variables, their contribution to the Berezinian appears in the denominator. Specifically, we obtain

$$-\text{tr} (\omega^{12} \omega^{12\dagger} + \omega^{21} \omega^{21\dagger}) \rightarrow \left(\prod_{p=1}^{k_1} \prod_{q=1}^{k_2} (s_{p1}^2 + s_{q2}^2) \right)^{-2} . \quad (\text{B11})$$

Collecting terms, we finally obtain the Berezinian (2.26).

Since Eq. (B3) implies that the metric tensor in the subspace of the radial coordinates is just the unit matrix, the radial part of the Laplacian has the form (2.27).

APPENDIX C: SEPARABILITY IN SUPERSPACE

Again, we write Δ_s in a more convenient form,

$$\Delta_s = \sum_{j=1}^2 \sum_{p=1}^{k_j} \left(\frac{\partial^2}{\partial s_{pj}^2} + \frac{\partial \ln J(s)}{\partial s_{pj}} \frac{\partial}{\partial s_{pj}} \right) . \quad (\text{C1})$$

We now evaluate the derivatives in Eq. (2.29) in analogy to the case of ordinary matrices. The calculation is somewhat more involved but still reasonably straightforward so that we only mention the intermediate result

$$\Delta_s \frac{\lambda(s)}{B_{k_1 k_2}(s^2)} = \frac{1}{B_{k_1 k_2}(s^2)} \sum_{j=1}^2 \sum_{p=1}^{k_j} \left(\frac{\partial^2}{\partial s_{pj}^2} + \frac{1}{s_{pj}} \frac{\partial}{\partial s_{pj}} - 4D_{pj} \right) \lambda(s) , \quad (\text{C2})$$

where $D_{pj} = S_{pj} + s_{pj}^2(S_{pj}^2 - T_{pj})$. Here, $S_{pj} = \tilde{S}_{pj} - \bar{S}_{pj}$ and $T_{pj} = \tilde{T}_{pj} - \bar{T}_{pj}$ with

$$\tilde{S}_{pj} = \sum_{\substack{q=1 \\ q \neq p}}^{k_j} \frac{1}{s_{pj}^2 - s_{qj}^2} \quad \bar{S}_{pj} = \sum_{q=1}^{k_{\chi(j)}} \frac{1}{s_{pj}^2 + s_{q\chi(j)}^2} \quad (\text{C3})$$

$$\tilde{T}_{pj} = \sum_{\substack{q=1 \\ q \neq p}}^{k_j} \frac{1}{(s_{pj}^2 - s_{qj}^2)^2} \quad \bar{T}_{pj} = \sum_{q=1}^{k_{\chi(j)}} \frac{1}{(s_{pj}^2 + s_{q\chi(j)}^2)^2} . \quad (\text{C4})$$

In the above, we have introduced the convention

$$\chi(j) = \begin{cases} 1 & \text{if } j = 2 \\ 2 & \text{if } j = 1 \end{cases} . \quad (\text{C5})$$

We now show that $\sum_{j=1}^2 \sum_{p=1}^{k_j} D_{pj} = 0$. According to the definition, we have

$$D_{pj} = \tilde{S}_{pj} + s_{pj}^2(\tilde{S}_{pj}^2 - \tilde{T}_{pj}) - \bar{S}_{pj} + s_{pj}^2(\bar{S}_{pj}^2 - 2\tilde{S}_{pj}\bar{S}_{pj} + \bar{T}_{pj}) . \quad (\text{C6})$$

For each j , the sum over p of the first two terms is zero in analogy to the case of ordinary matrices which was discussed in Appendix A. We are thus left with the remaining two terms which we denote by $-D'_{pj}$. Summing over p and j , some algebra leads to

$$\sum_{j=1}^2 \sum_{p=1}^{k_j} D'_{pj} = \sum_{p=1}^{k_1} \sum_{q=1}^{k_2} \frac{1}{s_{p1}^2 + s_{q2}^2} \left(\sum_{\substack{p'=1 \\ p' \neq p}}^{k_1} \frac{s_{q2}^2(s_{p1}^2 + s_{p'1}^2) + 2s_{p1}^2 s_{p'1}^2}{(s_{p1}^2 - s_{p'1}^2)(s_{p'1}^2 + s_{q2}^2)} + \sum_{\substack{q'=1 \\ q' \neq q}}^{k_2} \frac{s_{p1}^2(s_{q2}^2 + s_{q'2}^2) + 2s_{q2}^2 s_{q'2}^2}{(s_{q2}^2 - s_{q'2}^2)(s_{p1}^2 + s_{q'2}^2)} \right) . \quad (\text{C7})$$

Renaming $p \leftrightarrow p'$ and $q \leftrightarrow q'$ in the first and second term, respectively, shows that each of the two sums yields zero individually. This completes the proof.

APPENDIX D: ON BOUNDARY CONTRIBUTIONS IN THE CASE OF HERMITEAN SUPERMATRICES

The purpose of this Appendix is to clarify the role of some boundary contributions to the supersymmetric Itzykson-Zuber integral which arise in the case of Hermitean supermatrices. This discussion is not directly related to the main content of the present paper.

After we had computed the result (3.30) for the function $\eta(s, t)$, P.-B. Gossiaux pointed out to us that the structure of these contributions in the case of Hermitean supermatrices ought to be very similar. Indeed, this is true. A careful reexamination of the considerations following Eq. (B33) in Appendix B of Ref. 7 leads to additional terms very similar to the ones in Eq. (3.30). To clarify this, we calculate the function

$$\eta(s, t) = \frac{1}{B_k(s)} \det [\tilde{\eta}(s_{p1}, is_{q2}, t)]_{p,q=1,\dots,k} \quad (\text{D1})$$

in the case of Hermitean supermatrices for $k_1 = k_2 = k$. Here, the entries of the determinant are given by

$$\tilde{\eta}(s_{p1}, is_{q2}, t) = \frac{1}{2\pi t} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dr_1 dr_2}{r_1 - ir_2} \exp\left(-\frac{1}{2t} ((s_{p1} - r_1)^2 + (s_{p2} - r_2)^2)\right) . \quad (\text{D2})$$

We introduce polar coordinates $r_1 + ir_2 = \kappa \exp(i\vartheta)$ and $s_{p1} + is_{q2} = \mu \exp(i\psi)$ and find

$$\tilde{\eta}(s_{p1}, is_{q2}, t) = \frac{1}{2\pi t} \int_0^{\infty} d\kappa \exp\left(-\frac{1}{2t} (\kappa^2 + \mu^2)\right) \int_0^{2\pi} d\vartheta \exp(i\vartheta) \exp\left(\frac{\kappa\mu}{t} \cos(\vartheta - \psi)\right) . \quad (\text{D3})$$

The angular integration yields the modified Bessel function $I_1(\kappa\mu/t)$. The radial integration can then be performed using standard methods.²⁶ Collecting everything, we arrive at

$$\eta(s, t) = \frac{1}{B_k(s)} \det \left[\frac{1}{s_{p1} - is_{q2}} \left(1 - \exp\left(-\frac{s_{p1}^2 + s_{q2}^2}{2t}\right) \right) \right]_{p,q=1,\dots,k} . \quad (\text{D4})$$

As in the case of complex matrices, this function satisfies the diffusion equation

$$\frac{1}{2} \Delta_s \eta(s, t) = \frac{\partial}{\partial t} \eta(s, t) , \quad (\text{D5})$$

where the radial part Δ_s of the Laplacian is defined in Eq. (B19) of Ref. 7. The main results of Ref. 7 do not change due to these additional contributions.

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